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Application of Advanced Statistical Procedures for Adjustment of Results in Measurements of Displacements

Abstract: In this paper, the authors verified the formulated principles of the estimation of Gauss–Markov models in which estimated parameters **X** were random. For this purpose, methods for the prior definition of covariance matrix $\mathbf{C_X}$ for the estimated parameters were provided, which were used to determine the conditional covariance matrix of observation vector **L** and then estimate the most probable values of parameters $\hat{\mathbf{X}}$. Covariance matrix $\mathrm{Cov}(\hat{\mathbf{X}})$ obtained as a result of this estimation was used to define the limit values of the variance of these parameters.

Practical application of the proposed method for the Gauss–Markov model estimation for random parameters was illustrated on a fragment of a leveling network of points to determine the vertical displacements of a landslide surface.

Keywords: measurements in engineering surveying, Gauss–Markov model, diagonal covariance matrix

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1. Introduction

When handling an investment, one of the surveyor's basic tasks is to determine the coordinates with the greatest accuracy possible. According to the Regulation of the Minister of the Interior and Administration of November 9, 2011, on the technical standards for the performance of geodetic topographic and field surveys as well as the preparation and transfer of these surveys to the National Cartographic Documentation Center database, "during the performance of topographic and field surveys related to the handling of construction investments, a setting-out control may be used, adjusted for the geometrical structure and accuracy of the positions of its points to the type of investment and requirements specified in the construction documentation."

In engineering practice, the Gauss–Markov model is most commonly used to adjust observation results, taking into account a diagonal covariance matrix for the observed values. It is also possible to use the Gauss–Markov model to adjust the results of the surveys of double controls, taking into account the apparent observational equations (pseudo-observations) for the coordinates of the reference points. Wiśniewski [1] demonstrated that the same results were obtained both in the single-stage adjustment and multistage adjustment processes, which took into account the equations of the pseudo-observations. Baarda [2], Teunissen [3], Rao [4], and Cross [5] were preoccupied with the problem of the proper selection of weights. Kampmann [6] and Caspary [7] developed the estimation process based on the balanced accuracy of the observations. Kampmann [8] and Hekimoglu [9] presented original studies on the selection of weights. Different accuracies in the existing geodetic controls and measuring factors affecting the accuracy of determining the coordinates of the setting-out control points make the adjusted coordinates of the geodetic points random; i.e., covariance matrices should be formulated a priori for them.

In this paper, the principles of the estimation of Gauss–Markov model parameters and their variances were applied to a network of geodetic points at which the estimated parameters (the coordinates of the points) were random [10] and were verified on a particular example of a leveling network of points to determine the vertical displacements of a landslide surface.

2. Theoretical Bases for Gauss–Markov model (L, AX, H) with random parameters

Each observed value of λ may be defined by an observational equation in the general form:

$$\delta_{\lambda} + d(\lambda) = \lambda_{\text{observ}} - \lambda_{\text{approx}} \tag{1}$$

Variables in Formula (1):

- λ_{observ} – the random deviation of observed value,
- $d(\lambda)$ – the differential of a function describing the variability of analyzed component λ with respect to the coordinates of the points of the geodetic network defining that component,
- λ_{approx} – the approximate value of the analyzed component, determined subject to the approximate coordinates of the points of the geodetic network.

If **L** is a vector of random variables equal to $(\lambda_{observ} - \lambda_{approx})$ and the average value of vector **L** can be written as $E(\mathbf{L}) = \mathbf{AX}$ (where **X** is the vector of unknown parameters), then matrix **A** represents the matrix of the coefficients defined by the values of partial derivatives occurring in differentials $d(\lambda)$. Vector of unknown parameters **X** also represents a random variable for which it is possible to determine a priori covariance matrix $\mathbf{C_X}$.

Assuming that matrix $\mathbf{H} = \mathbf{P}^{-1}$, the conditional covariance matrix of observation vector **L** can be determined by the following dependence:

$$V(\mathbf{L}) = E[V(\mathbf{L}/\mathbf{X})] + V[E(\mathbf{L}/\mathbf{X})] = \mathbf{H} + V(\mathbf{AX}) = \mathbf{H} + \mathbf{AC_X A}^T \tag{2}$$

Taking into account the conditional covariance matrix of observation vector **L**, the square form of F for random deviations takes the following form:

$$F = [(\mathbf{L} - \mathbf{AX})^T(\mathbf{H} + \mathbf{AC_X A}^T)^{-1}(\mathbf{L} - \mathbf{AX})] = \min \tag{3}$$

After the transformations (with the assumption for the minimum of Formula (3)), an alternative formula for calculating the estimator of vector $\hat{\mathbf{X}}$ was obtained:

$$\hat{\mathbf{X}} = (\mathbf{C_X}^{-1} + \mathbf{A}^T\mathbf{H}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{H}^{-1}\mathbf{L} \tag{4}$$

After a full analysis of the variance, a functional relationship was obtained that represents the estimated variance $\sigma^2(\hat{X}_i)$ of the analyzed parameter $(\hat{X}_i)$ and the tested variance $\sigma^2(X_i)$ of this parameter, taking into account $k = n - u$ degrees of freedom determined by the chi-square (χ^2) method according to the following formula:

$$\chi^2 = \frac{k \cdot \sigma^2(\hat{X}_i)}{\sigma^2(X_i)} \tag{5}$$

Finally, a relationship between the estimated variance and tested variance in conjunction with distribution quantile (χ^2) was obtained; i.e.:

$$P = \left[\frac{k \cdot \sigma^2(\hat{X})}{\sigma^2(X)} > \chi^2(\alpha_i;\, k) \right] = 1 - \alpha \tag{6}$$

where the condition for the tested variance for the confidence level of $(1 - \alpha)$ can be written using the following formula:

$$\sigma^2(X) \le \frac{k \cdot \sigma^2(\hat{X})}{\chi^2(\alpha_i;\, k)} \Leftrightarrow \sigma(X) \le \sqrt{\frac{k}{\chi^2(\alpha_i;\, k)}}\sigma(\hat{X}) \tag{7}$$

The above formula can be used to set the limit values for the standard deviations of the estimated parameters.

3. Numerical Example of Applying Gauss–Markov Model with Random Parameters

The numerical example concerns a fragment of a leveling network of points, which consists of four benchmarks representing the surface of a landslide (as illustrated in Figure 1).

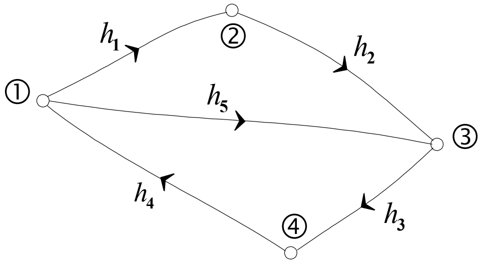

Fig. 1. Fragment of leveling network of points

Based on the observations of the first periodic measurement, the most probable heights of these benchmarks and their covariance matrix were obtained. As demonstrated in Figure 1, the values of these parameters are as follows:

$$\hat{\mathbf{X}}_{\mathrm{I}} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 2.4002 \\ 3.4004 \\ 2.4000 \\ 3.3980 \end{bmatrix}, \quad \mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}}) = \begin{bmatrix} 1.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & 2.5 & -0.5 & -1.0 \\ -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & -1.0 & -0.5 & 2.5 \end{bmatrix}.$$

During the second periodic measurement, five height differences h_i were observed, each of which was observed from Stand 1 of the leveler. The results of these observations along with the calculated height differences and the free terms for the observational equations are presented in the form of one-column matrices:

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} 1.0024 \\ -1.0052 \\ 1.0019 \\ -0.9992 \\ -0.0006 \end{bmatrix} [\mathrm{m}], \quad \Delta\mathbf{z} = \begin{bmatrix} \Delta z_{12} \\ \Delta z_{23} \\ \Delta z_{34} \\ \Delta z_{41} \\ \Delta z_{13} \end{bmatrix} = \begin{bmatrix} 1.0002 \\ -1.0004 \\ 0.9980 \\ -0.9978 \\ -0.0002 \end{bmatrix} [\mathrm{m}],$$

$$\mathbf{L}_{\mathrm{II}} = \mathbf{H} - \Delta\mathbf{z} \begin{bmatrix} 2.2 \\ -4.8 \\ 3.9 \\ -1.4 \\ -0.4 \end{bmatrix} [\mathrm{mm}].$$

The observational equations (written in matrix form $\boldsymbol{\delta} + \mathbf{A}\hat{\mathbf{X}}_{\mathrm{II}} = \mathbf{L}_{\mathrm{II}}$) take the following explicit form:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} dz_1 \\ dz_2 \\ dz_3 \\ dz_4 \end{bmatrix} = \begin{bmatrix} 2.2 \\ -4.8 \\ 3.9 \\ -1.4 \\ -0.4 \end{bmatrix} [\mathrm{mm}].$$

It is evident that $R(A) = 3$, as each column of matrix $\mathbf{A}$ is a linear combination of the remaining three columns, then matrix $\mathbf{A}$ (representing the system of equations with four unknowns) has a defect of $d = m - R(A) = 4 - 3 = 1$. According to the observation program, the covariance matrix for the observed values $\mathbf{H} = \mathbf{P}^{-1}$, is a unit matrix of the size (5×5).

Having taken into account the above assumptions and numerical data, the formula to calculate the estimator of vector $\hat{\mathbf{X}}_{\mathrm{II}}$ will take the following form:

$$\hat{\mathbf{X}}_{\mathrm{II}} = [\mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}})^{-1} + \mathbf{A}^T\mathbf{H}^{-1}\mathbf{A}]^{-1}\mathbf{A}^T\mathbf{H}^{-1}\mathbf{L}_{\mathrm{II}} \Leftrightarrow \hat{\mathbf{U}}_{\mathrm{II}}$$

The values of the components of each matrix contained in the above formula are as follows:

$$\mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}})^{-1} = \begin{bmatrix} 1.75 & 1 & 1.25 & 1 \\ 1 & 1.1429 & 1 & 0.8571 \\ 1.25 & 1 & 1.75 & 1 \\ 1 & 0.8571 & 1 & 1.1429 \end{bmatrix},$$

$$\mathbf{A}^T\mathbf{H}^{-1}\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

$$[\mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}})^{-1} + \mathbf{A}^T\mathbf{H}^{-1}\mathbf{A}]^{-1} = \begin{bmatrix} 0.2111 & 0 & -0.011 & 0 \\ 0 & 0.3438 & 0 & -0.094 \\ -0.011 & 0 & 0.2111 & 0 \\ 0 & -0.094 & 0 & 0.3438 \end{bmatrix},$$

$$[\mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}})^{-1} + \mathbf{A}^T\mathbf{H}^{-1}\mathbf{A}]^{-1}\mathbf{A}^T\mathbf{H}^{-1} = \begin{bmatrix} -0.211 & -0.011 & 0.011 & 0.211 & -0.222 \\ 0.344 & -0.344 & -0.094 & 0.094 & 0 \\ 0.011 & 0.211 & -0.211 & -0.011 & 0.222 \\ -0.094 & 0.094 & 0.344 & -0.344 & 0 \end{bmatrix}.$$

After the last matrix has been multiplied by the free term matrix, a parameter estimator is obtained that defines the vertical displacement of the benchmarks; i.e.:

$$\hat{\mathbf{X}}_{\mathrm{II}} = \hat{\mathbf{U}}_{\mathrm{II}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.57 \\ +1.91 \\ -1.89 \\ +1.17 \end{bmatrix} \text{[mm]}.$$

The random corrections applied to the observed height differences and the variance for the estimated model take the following values:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix} = \begin{bmatrix} 2.2 \\ -4.8 \\ 3.9 \\ -1.4 \\ -0.4 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -0.57 \\ 1.91 \\ -1.89 \\ 1.17 \end{bmatrix} = \begin{bmatrix} -0.28 \\ -1.01 \\ 0.85 \\ 0.34 \\ 0.91 \end{bmatrix} \text{[mm]},$$

$$\sigma^2_{\mathrm{II}} = \frac{2.757}{5-3} = 1.38\ \mathrm{mm}^2.$$

The covariance matrix for the estimated vertical displacements of the points takes the following values:

$$\mathrm{Cov}(\hat{\mathbf{U}}_{\mathrm{II}}) = \sigma_{\mathrm{II}}^2[\mathrm{Cov}(\hat{\mathbf{X}}_{\mathrm{I}})^{-1} + \mathbf{A}^T\mathbf{H}^{-1}\mathbf{A}]^{-1} = 1.38 \cdot \begin{bmatrix} 0.2111 & 0 & -0.011 & 0 \\ 0 & 0.3438 & 0 & -0.094 \\ -0.011 & 0 & 0.2111 & 0 \\ 0 & -0.094 & 0 & 0.3438 \end{bmatrix}.$$

Based on the covariance matrix presented above, the standard deviations for the determined vertical displacements of the benchmarks will be determined; i.e.:

$$\sigma(\hat{\mathbf{U}}_{\mathrm{II}}) = \begin{bmatrix} \sigma(u_1) \\ \sigma(u_2) \\ \sigma(u_3) \\ \sigma(u_4) \end{bmatrix} = \begin{bmatrix} 0.54 \\ 0.69 \\ 0.54 \\ 0.69 \end{bmatrix}.$$

Having determined the quantiles of the chi-square distribution for the k = 2 degrees of freedom and the confidence level of $(1 - \alpha) = 0.90$ it follows that coefficient $\sqrt{k/\chi^2(\alpha;\ k)} = 3.1$. Thus, the limit values of the standard deviations defined according to Formula (7) for the determined vertical displacements take the following values:

$$\begin{bmatrix} \sigma_G(u_1) \\ \sigma_G(u_2) \\ \sigma_G(u_3) \\ \sigma_G(u_4) \end{bmatrix} = 3.1 \cdot \begin{bmatrix} 0.54 \\ 0.69 \\ 0.54 \\ 0.69 \end{bmatrix} = \begin{bmatrix} 1.67 \\ 2.14 \\ \mathbf{1.67} \\ 2.14 \end{bmatrix} \Leftrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.57 \\ +1.91 \\ \mathbf{-1.89} \\ +1.17 \end{bmatrix}.$$

A comparison of the vertical displacements of the benchmarks to the limit values of their standard deviations proves that, at a confidence level of $1 - \alpha = 0.90$, only Benchmark (3) demonstrates the significant vertical movements observed in the two analyzed periodic measurements.

In order to compare the adjustment results obtained from the Gauss–Markov model with random parameters and a traditional method of adjusting a leveling network, the system of observational equations defined by formula $\boldsymbol{\delta}_2 + \mathbf{A}\hat{\mathbf{X}} = \mathbf{L}_{\mathrm{II}}$ (i.e., in explicit form) will be considered once again:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} dz_1 \\ dz_2 \\ dz_3 \\ dz_4 \end{bmatrix} = \begin{bmatrix} 2.2 \\ -4.8 \\ 3.9 \\ -1.4 \\ -0.4 \end{bmatrix} \text{ [mm].}$$

As was previously found, matrix **A** has a defect of $d = m - R(A) = 1$. Therefore, solving this system of equations by ordinary inverse of matrix $(\mathbf{A}^T\mathbf{A})$ (which represents a system of normal equations) is impossible, since the determinant of this matrix is equal to zero.

In order to solve this system of observational equations, it must be freed from the defect. It may be performed numerically by deleting the columns equal in number to the defect, which is one column to be deleted in this case – the fourth one. In a geometric interpretation, this will mean that Benchmark (4) that corresponds to the deleted column explicitly defines the reference level in the adjustment process. Such a model adopted for the adjustment of the observation results is expressed by the following system of observational equations:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} dz_1 \\ dz_2 \\ dz_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 8 \\ 16 \\ 0 \end{bmatrix} \text{[mm]}, \quad dz_4 = 0.$$

The matrices defined for this model will be denoted by the lower index 2; therefore, the above system of equations can be written in the following matrix form:

$$\boldsymbol{\delta}_2 + \mathbf{A}\mathbf{X}_{\text{II}} = \mathbf{L}_{\text{II}}.$$

With the earlier adoption of matrix $\mathbf{H} = \mathbf{P}^{-1} = \mathbf{I}$, the above system of equations is solved according to the following matrices:

$$\mathbf{A}_2^T\mathbf{A}_2 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad (\mathbf{A}_2^T\mathbf{A}_2)^{-1} = \frac{1}{8}\begin{bmatrix} 5 & 4 & 3 \\ 4 & 8 & 4 \\ 3 & 4 & 5 \end{bmatrix},$$

$$\mathbf{A}_2^{\oplus} = (\mathbf{A}_2^T\mathbf{A}_2)^{-1}\mathbf{A}_2^T = \frac{1}{8}\begin{bmatrix} -1 & -1 & -3 & 5 & -2 \\ 4 & -4 & -4 & 4 & 0 \\ 1 & 1 & -5 & 3 & 2 \end{bmatrix}.$$

The values of the corrections to the heights of the benchmarks (determined based on the first survey) are calculated from the product of matrix $A_2^{\oplus}L$; i.e.:

$$\hat{\mathbf{X}}_2 = \mathbf{A}_2^{\oplus}\mathbf{L} = \begin{bmatrix} -1.9 \\ +0.8 \\ -3.4 \end{bmatrix} \text{[mm]} \Rightarrow \begin{bmatrix} dz_1 \\ dz_2 \\ dz_3 \\ dz_4 \end{bmatrix} = \begin{bmatrix} -0.0019 \\ +0.0008 \\ -0.0034 \\ 0 \end{bmatrix} \text{[m]} \Leftrightarrow \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} = \begin{bmatrix} -1.9 \\ +0.8 \\ -3.4 \\ 0 \end{bmatrix}.$$

The above calculated values of corrections dz_i represent the apparent displacements $\bar{u}_i$ of the benchmarks, since they were determined with the condition of $dz_4 = 0$.

For comparison, the vertical displacements of the benchmarks obtained from the first model were presented as follows:

$$\hat{\mathbf{U}}_{\mathrm{II}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.57 \\ +1.91 \\ -1.89 \\ +1.17 \end{bmatrix}.$$

The vector of random deviations $\boldsymbol{\delta}$ takes the following values:

$$\boldsymbol{\delta}_2 = \mathbf{L} - \mathbf{A}_2\hat{\mathbf{X}}_2 = \begin{bmatrix} -0.6 \\ -0.6 \\ 0.5 \\ 0.5 \\ 1.1 \end{bmatrix} [\mathrm{mm}], \quad \text{whence} \quad \sigma_2^2 = \frac{2.314}{5-3} = 1.16\ \mathrm{mm}^2.$$

The covariance matrix of vector $\hat{\mathbf{X}}_2$ takes the following values:

$$\mathrm{Cov}\hat{\mathbf{X}}_2 = = \hat{\sigma}_2^2(\mathbf{A}_2^T\mathbf{A}_2)^{-1} = \frac{\hat{\sigma}_2^2}{8}\begin{bmatrix} 5 & 4 & 3 \\ 4 & 8 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Based on the above covariance matrix and variance $\sigma_2^2 = 1.16\ \mathrm{mm}^2$, the standard deviations were defined for the determined apparent displacements of the benchmarks; i.e.:

$$\sigma(\bar{\mathbf{U}}_2) = \begin{bmatrix} \sigma(\bar{u}_1) \\ \sigma(\bar{u}_2) \\ \sigma(\bar{u}_3) \end{bmatrix} = \begin{bmatrix} 0.85 \\ 1.08 \\ 0.85 \end{bmatrix} \quad \text{and} \quad \sigma(\bar{u}_4) = 0.$$

For comparison, the standard deviations for the vertical displacements obtained from the first model were presented as follows:

$$\sigma(\mathbf{U}_{\mathrm{II}}) = \begin{bmatrix} \sigma(u_1) \\ \sigma(u_2) \\ \sigma(u_3) \\ \sigma(u_4) \end{bmatrix} = \begin{bmatrix} 0.54 \\ 0.69 \\ 0.54 \\ 0.69 \end{bmatrix}.$$

By comparing the results of adjusting the leveling network using the Gauss–Markov model with random parameters and the second model without a defect, a general conclusion may be drawn that the displacements and their obtained standard deviations from the first model have smaller values than in the second model.

4. Conclusions

The algorithm for the estimation of the Gauss–Markov model with random parameters to adjust the results of periodic surveys of the networks of points proposed by the authors allows us to determine the most likely coordinates of the points and components of their displacements as well as the standard deviations for these parameters.

The demonstrated numerical example for a fragment of the leveling network of periodically observed benchmarks illustrates various stages of implementation of the G-M model estimation algorithm in detail for the current survey, taking into account the covariance matrix for the heights of the benchmarks specified in the process of adjusting the output measurement.

An analysis of the results of adjusting the leveling network using the Gauss–Markov model with random parameters and the second model without a defect allows us to formulate a general conclusion that the displacements and their obtained standard deviations from the first model will always have smaller values than in the second model.

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Weryfikacja zaawansowanych procedur statystycznych do wyrównywania wyników w pomiarach przemieszczeń

Streszczenie: W artykule autorzy poddali weryfikacji sformułowane zasady estymacji modeli Gaussa–Markowa [10], w których szacowane parametry **X** miały charakter losowy. W tym celu zostały podane sposoby określania *a priori* macierzy kowariancji $\mathbf{C}_X$ dla estymowanych parametrów, które zostały wykorzystane do wyznaczenia macierzy kowariancji warunkowych wektora obserwacji **L**, a następnie do estymacji najbardziej prawdopodobnych wartości parametrów $\hat{\mathbf{X}}$. Uzyskana w wyniku tej estymacji macierz kowariancji Cov($\hat{\mathbf{X}}$) została wykorzystana do ustalenia granicznych wartości wariancji tych parametrów. Zastosowanie proponowanego sposobu estymacji modelu Gaussa–Markova do parametrów losowych zostało zilustrowane na przykładzie fragmentu niwelacyjnej sieci punktów przeznaczonej do wyznaczania pionowych przemieszczeń powierzchni osuwiska.

Słowa kluczowe: pomiary w geodezji inżynieryjnej, model Gaussa–Markova, diagonalna macierz kowariancyjna